Efficient object classification using the Euler characteristic

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to obtain the degree of
Licenciado en matemáticas

División de Ciencias Naturales y Exactas
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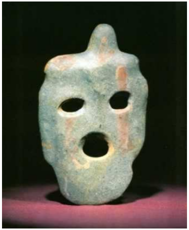
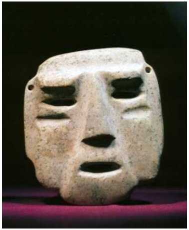
16:00h, 28 - MAY - 2018

Main goal: understand the morphology of pre-Columbian masks





- Between 1978 and 1982 several offerings were excavated in Templo Mayor (Mexico City).
- Among the artifacts, 162 pre-Columbian masks were found.
- It is unclear how many and which cultures are exactly represented.
- The actual classifications are prone to subjectivities.
- **With Topological Data Analysis (TDA) tools we pretend to provide a more objective understanding of these masks' morphology.**

Acknowledgements

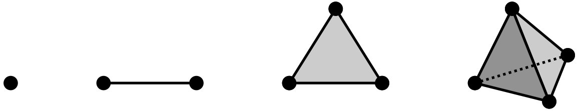
- Antonio Rieser, for his constant advising throughout the thesis.
- Mario Canul, for teaching me on how to computationally handle the data.
- Red Temática CONACYT de Tecnologías Digitales para la Difusión del Patrimonio Cultural (RedTDPC) and the Instituto Nacional de Antropología e Historia (INAH) for providing the data that made this thesis.
- Veranos de Investigación Científica AMC 2016.

The thesis is quite modular

1. Simplicial homology
2. Euler Characteristic Graph (ECG)
3. Support Vector Machines (SVM)
4. Unsupervised SVMs
5. Archaeological data
6. Conclusions and future work

- Take $\mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^d$ vertices in general position.
- The ***d*-simplex** is the convex hull of these vertices.

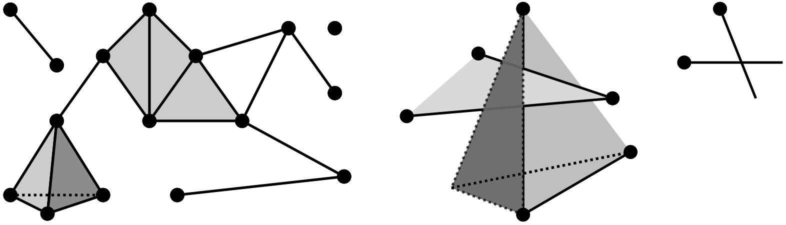
$$S_d := \left\{ \sum_{i=0}^{d} \lambda_i \mathbf{v}_i \ : \ \lambda_i \geq 0, \ \sum_{i=0}^{d} \lambda = 1 \right\}$$



d-simplices for $d = 0, 1, 2, 3$

- We'll denote simplices as $\sigma = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d)$.
- Refer to $\tau = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{d-1})$ as a **face** of σ.

- A **simplicial complex** is a collection of nicely glued simplices.
  - Every simplex comes will all its faces.
  - Any two neighboring simplices share a whole face.

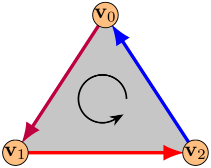


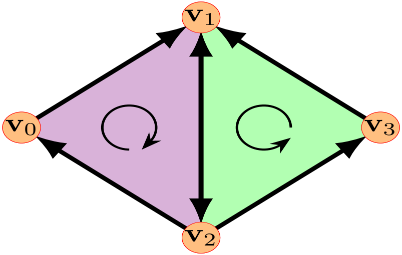
(a) Good **(b)** Bad

- The **dimension** of a dimension is the dimension of its highest dimensional simplex.

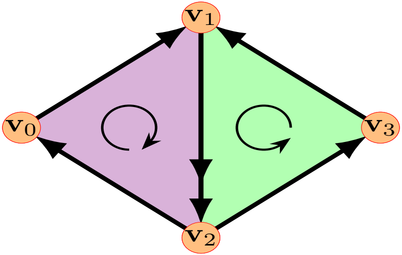
Orientatation and compatibility

- $\sigma = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d)$ denotes **one of two possible** different orientations.





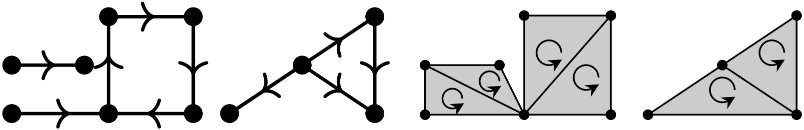
(a) Compatibly oriented triangles



(b) Incompatibility

Chains and Homomorphisms

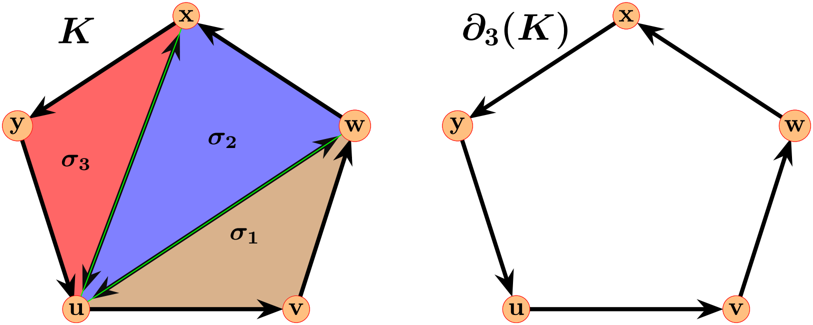
- Group of ***q*-chains** $C_q = \left\{ \sum_{k=1}^{r} \lambda_k \sigma_k \ : \ \lambda_k \in \mathbb{Z},\ \sigma_k \text{ un } q\text{-simplejo} \right\}$.
- $-\sigma$ has **opposite** orientation to σ.
- Define an **homomorphism** φ by defining for every simplex and then extend it **linearly**.
- Just have to verify that $\varphi(-\sigma) = -\varphi(\sigma)$.
- $\varphi(c) = \sum \lambda_k \varphi(\sigma_k)$



(a) 1-chain **(b)** 2-chain

Boundary homomorphism $\partial_q : C_q \to C_{q-1}$

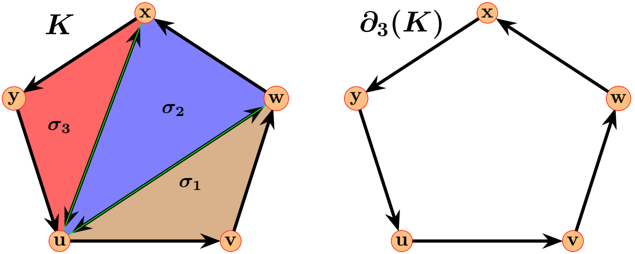
- $\partial\sigma = \sum_{i=0}^{q}(-1)^i(\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$.
- $\partial^2 = \partial_q \circ \partial_{q-1} = 0$.



$\partial_2(\partial_3(K)) = (\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{x} - \mathbf{w}) + (\mathbf{y} - \mathbf{x}) + (\mathbf{u} - \mathbf{y}) = \mathbf{0}$.

Cycles, boundaries and homology

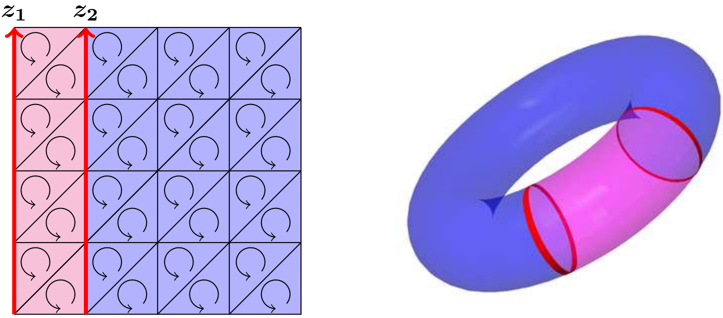
- $Z_q(K) = \ker \partial_q$ denotes the group of q-**cycles**
- $B_q(K) = \mathrm{im}\, \partial_{q+1}$ denotes the group of q-**boundary cycles**
- $H_q(K) = Z_q(K)/B_q(K)$ is the q-th group of **homology**.
- $H_q(K) \simeq F \oplus T$ por ser grupo abeliano finitamente generado.
- $\beta_q(K) = \dim(H_q(K))$ is the q-th **Betti number.**



$\partial_2(\partial_3(K)) = (\mathbf{v} - \mathbf{u}) + (\mathbf{w} - \mathbf{v}) + (\mathbf{x} - \mathbf{w}) + (\mathbf{y} - \mathbf{x}) + (\mathbf{u} - \mathbf{y}) = \mathbf{0}$.

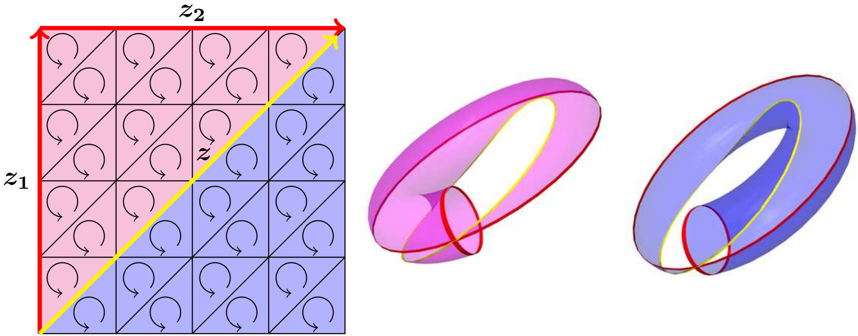
$H_q(K) = Z_q(K)/B_q(K)$

- Two cycles are **homological** if their difference is a boundary cycle.



Red cycles are homological in the torus

$H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ y $\beta_1(T) = 2$



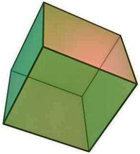
$z_1 + z_2 + z$ is a boundary cycle defined by the pink 2-chain

- In general, β_q records the **number** of homologically different **holes**!

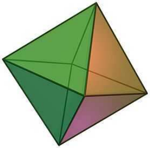
The Euler characteristic

- Assume $K = \bigcup_{q=0}^{d} V_q(K)$.
- Its **Euler characteristic** is defined as $\chi(K) = \sum_{q=0}^{d} (-1)^q |V_q(K)|$.
- Due to the **Euler-Poincaré** formula, we have that

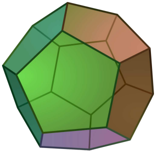
$$\chi(K) = \sum_{q=0}^{d} (-1)^q \beta_q(K).$$



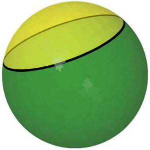
(a) $8 - 12 + 6 = 2$



(b) $6 - 12 + 8 = 2$

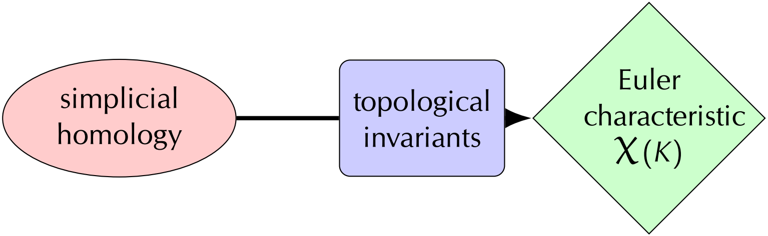


(c) $20 - 30 + 12 = 2$



$\chi(S^2) = 1 - 0 + 1 = 2$

Where are we? (I)

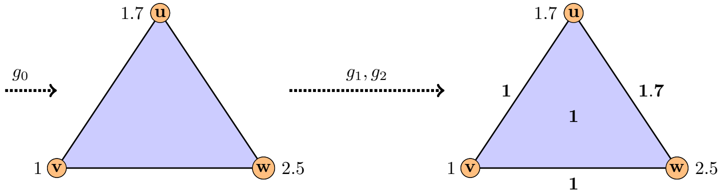


- Idea proposed by Richardson and Weirman in [5]
- Fix a filter function g for every vertex and then extend it to the rest of q-simplices:

$$g_q(\ \{v_0, v_1, \ldots, v_q\}\ ) = \min_{0 \le i \le k} \{\ g(v_i)\ \}$$

$g_q : V_q \to [a, b]$ $\quad$ q-simplex

A function $g : V_0 \to [a, b]$
set of vertices V_0;
fixed interval$[a, b]$.



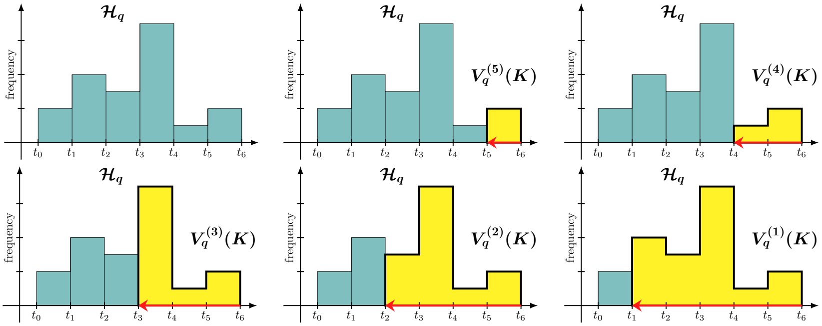
ECG: Thresholds

- T uniformly spaced thresholds $a = t_0 < t_1 < \ldots < t_T = b$.
- For each t_i we define $V_q^{(i)} = \{\sigma \in V_q \ : \ g(\sigma) > t_i\}$.
- $\chi^{(i)} = \sum_{q=0}^{d}(-1)^q |V_q^{(i)}|$ is the **Euler characteristic at the *i*-th threshold.**

Filter: $1/(10\text{th nearerst neighbor})$ (KNN)

Computing the ECG: $O(V)$

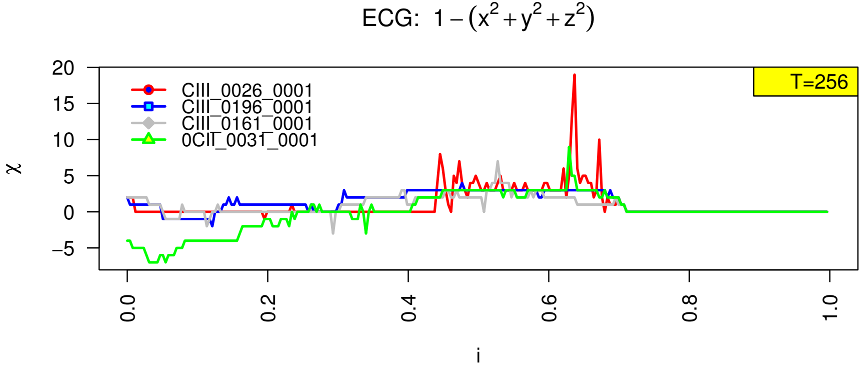
- The computation is efficient by using a histogram and bucket-sort.
- The Euler Characteristic is a constant-time operation.



Bucket-sort to compute the ECG

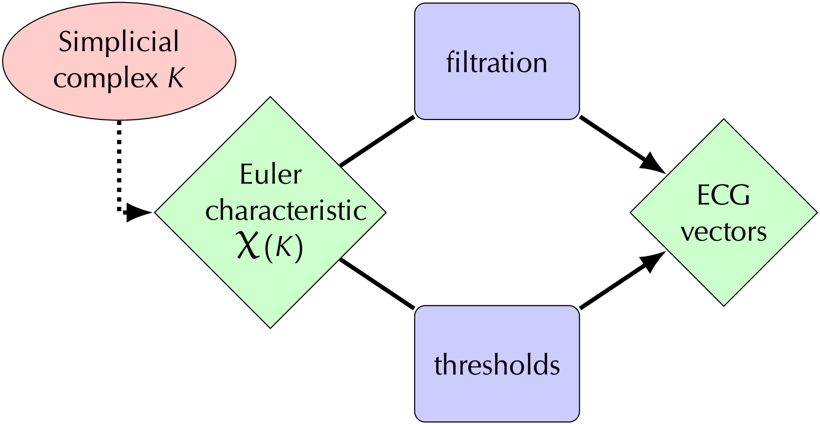
The ECG

- the **Euler Characteristic Graph** is the graph obtained as χ_i vs. t_i.



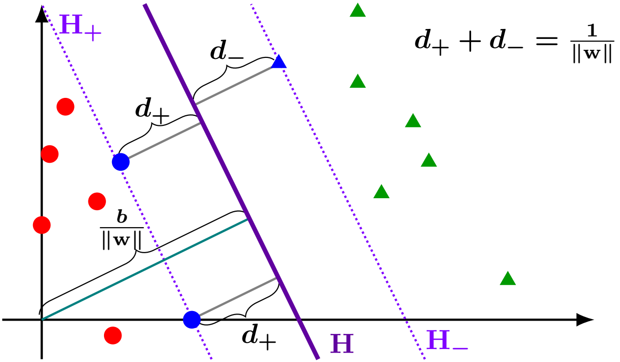
- Each ECG can be thought as a **vector** $(\chi_0, \chi_1, \ldots, \chi_{T-1}) \in \mathbb{R}^T$.

Where are we? (II)



SVM: linear separable case

- We have n labeled vectors $\{\mathbf{x}_i, y_i\}_{i=1}^{n} \subset \mathbb{R}^d$ with $y_i \in \{-1, +1\}$.
- Must split the labels with the *best possible* **hyperplane H**.
- **H** is defined by a normal vector **w** and a scalar b.
- $\mathbf{H} = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{w} \rangle + b = 0\}$.



Constrained optimization

- In general for C^1 functions we want to solve

$$\begin{aligned} \min \quad & f(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \\ \text{where} \quad & c_i(\mathbf{x}) = 0, & i \in E, \\ & c_i(\mathbf{x}) \geq 0, & i \in I. \end{aligned}$$

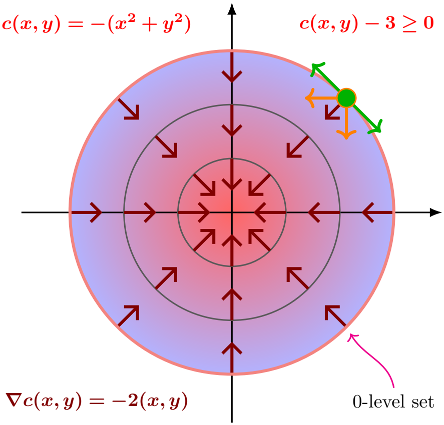
- If $I = \varnothing$, then we can use Lagrange multipliers.
- Define the subset of **active constraint indexes**

$$\mathcal{A}(\mathbf{x}) := \{i \in E \cup I \,:\, c_i(\mathbf{x}) = 0\}$$

Only active constraints matter

- Consider the set of **feasible directions** which are obtained by **linearized** constraints.

$$F(\mathbf{x}) = \{\mathbf{s} \,:\, \mathbf{s} \neq 0, \langle \mathbf{s}, \nabla c_i(\mathbf{x}) \rangle = 0, i \in E, \langle \mathbf{s}, \nabla c_i(\mathbf{x}) \rangle \geq 0, i \in I \cap \mathcal{A}(\mathbf{x})\}$$



Generalizing Lagrange multipliers

Theorem (Karush-Kuhn-Tucker Conditions)

If $\mathbf{x}$ *is a local minimum and certain regularity holds at* $\mathbf{x}$*, then there are multipliers* $\{\alpha_i\}_i$ *such that the following system is satisfied:*

$$\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x},\boldsymbol{\alpha}) = 0, \textit{ where } \mathcal{L}(\mathbf{x},\boldsymbol{\alpha}) = f(\mathbf{x}) - \sum_{i\in E\cup I} \alpha_i c_i(\mathbf{x}); \tag{5.1a}$$

$$c_i(\mathbf{x}) = 0,\ i \in E; \tag{5.1b}$$

$$c_i(\mathbf{x}) \geq 0,\ i \in I; \tag{5.1c}$$

$$\alpha_i \geq 0,\ i \in I; \tag{5.1d}$$

$$\alpha_i c_i(\mathbf{x}) = 0,\ \forall i. \tag{5.1e}$$

- If the optimization problem is convex, then KKT conditions are also sufficient.

SVM as constrained optimization

$$\min_{(\mathbf{w},b)\in\mathbb{R}^d\times\mathbb{R}} \quad f(\mathbf{w},b) := \frac{1}{2}||\mathbf{w}||^2,$$

$$\text{such that} \quad c_i(\mathbf{w},b) := y_i(\langle \mathbf{x}_i, \mathbf{w}\rangle + b) \geq 1 \text{ for all } i = 1,\ldots,n.$$

- With the **KKT conditions** and **Wolfe dual** the SVM problem is:

$$\max_{\alpha_i\geq 0} \sum_{1\leq i\leq n} \alpha_i - \frac{1}{2}\sum_{1\leq i,j\leq n} \alpha_i\alpha_j y_i y_j \langle \mathbf{x}_i, \mathbf{x}_j\rangle,$$

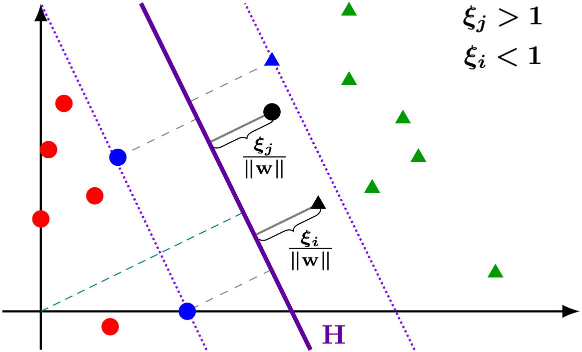
where

$$\mathbf{w} = \sum_{i=1}^{n} \alpha_i y_i \mathbf{x}_i.$$

$$b = y_i - \langle \mathbf{w}, \mathbf{x}_i\rangle \text{ for some } \alpha_i > 0.$$

SVM: non-separable linear case

- Suppose now there are mistakes $\xi_i \geq 0$ for each point.
- It is a actual mistake when $\xi_i > 1$.
- We must minimize then $\sum_i \xi_i$.



Same optimization problem

$$\min_{(\mathbf{w},b,\boldsymbol{\xi})\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d} f(\mathbf{w},b,\boldsymbol{\xi}) := \frac{||\mathbf{w}||^2}{2} + C\left(\sum_{i=0}^{n}\xi_i\right)^k, \quad C>0,\ k\geq 1,$$

$$\text{such that } y_i(\langle \mathbf{x}_i, \mathbf{w}\rangle + b) + \xi_i \geq 1 \text{ for all } i = 1,\dots,n$$

$$\text{and } \xi_i \geq 0.$$

- With $k = 1$, KKT and Wolfe again we obtain:

$$\max_{0\leq\alpha_i\leq C} \sum_{1\leq i\leq n}\alpha_i - \frac{1}{2}\sum_{1\leq i,j\leq n}\alpha_i\alpha_j y_i y_j \langle \mathbf{x}_i, \mathbf{x}_j\rangle.$$

- To test new points **x**, we simply do

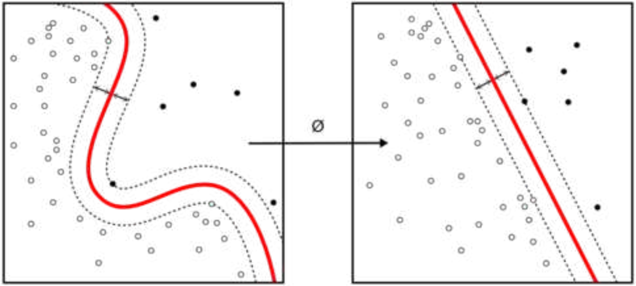
$$\text{class}(\mathbf{x}) = \text{sgn}(\langle \mathbf{w}, \mathbf{x}\rangle)$$

SVM: Nonlinear case & kernelization

- $\Phi : \mathbb{R}^d \to \mathcal{H}$, where $\mathcal{H}$ is a high-dimensional Hilbert space where we solve linearly the SVM.
- We must use **kernel** functions $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

$$K(\mathbf{x}, \mathbf{y}) = \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}}$$

- If we know K explicitly, we do not need to know Φ or $\mathcal{H}$.



Mercer theorem and RKHS's

Theorem (Mercer's condition)

For a compact subset $C \subset \mathbb{R}^d$ and given continuous function $K : C \times C \to \mathbb{R}$ there exists a mapping Φ, and a Hilbert space $\mathcal{H}$ such that

$$K(\mathbf{x}, \mathbf{y}) = \langle \Phi(\mathbf{x}), \Phi(\mathbf{y}) \rangle_{\mathcal{H}} \quad \forall\, \mathbf{x}, \mathbf{y} \in C \tag{5.2}$$

if and only if for any $L_2(C)$ function $g : C \to \mathbb{R}$ (that is, g^2 is Lebesgue-integrable on C) the following inequality holds

$$\int_C \int_C K(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) g(\mathbf{y}) \, \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} \geq 0. \tag{5.3}$$

- The proof uses machinery from functional analysis and Reproducing Kernel Hilbert Spaces.

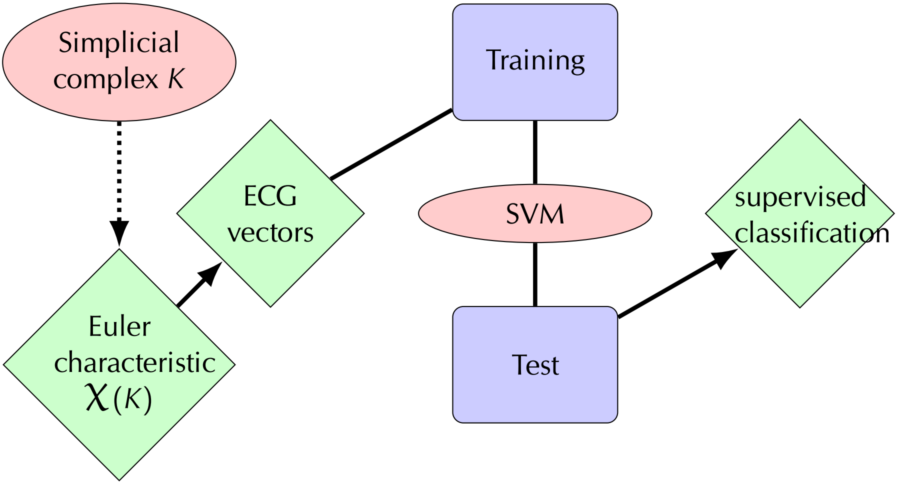
Multiclass SVM: All-vs-All (AvA)

- Solve $\binom{m}{2}$ different SVMs, one per possible pair of labels.
- The (j,k)-th SVM will relabel the training data as $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$ where $y_i = +1$ if $l_i = j$, $y_i = -1$ if $l_i = k$ and $y_i = 0$ otherwise.
- Produce $\binom{m}{2}$ test functions

$$t_{j,k}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}_{j,k} \rangle + b_{j,k}.$$

- Max-votes strategy.

Where are we? (III)



Maximum Margin Problem (MMP)

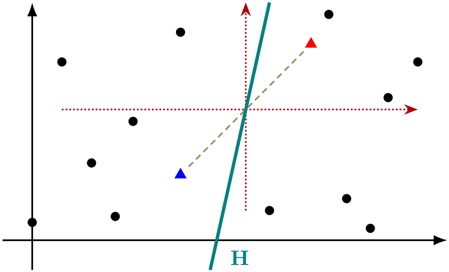
- Unsupervised classification, as we assume n points $\mathbf{x}_i$ with their labels y_i unknown.
- The **Maximum Margin Problem (MMP)** approach:
  - Consider all the 2^{n-1} possible different labellings.
  - Solve 2^{n-1} SVMs, one per labelling.
  - Record 2^{n-1} margins and pick the largest one.
- The MMP procedure is extremely expensive.
- A given hyperplane $\mathbf{H}(\mathbf{w},b)$ will label the data

$$\mathrm{class}(\mathbf{x}) = \mathrm{sgn}(\langle \mathbf{w}, \mathbf{x} \rangle + b)$$

- We might just look for a hyperplane *good enough*.

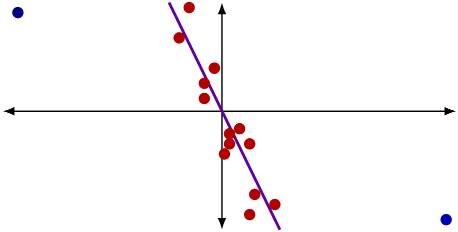
Furthest Hyperplane Problem (FHP)

- Assume that the hyperplane goes through the origin.
- Translate and rescale accordingly.
- To solve MMP, we must solve efficiently the FHP $\binom{n}{2}$ times.



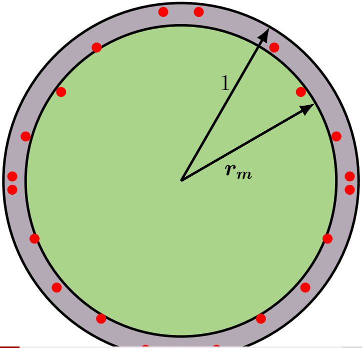
Approximate FHP

- We first want to compute a **w** with norm 1 such that it maximizes mean($\langle \mathbf{w}, \mathbf{x}_i \rangle$).
- Observe that the **first singular vector** is the answer.
- This approach might fail if there are outliers.
- Re-weight the distances: penalize the points close to the singular vector.
- Compute the first singular vector again and iterate.
- Consider the Gaussian combination of all of them.

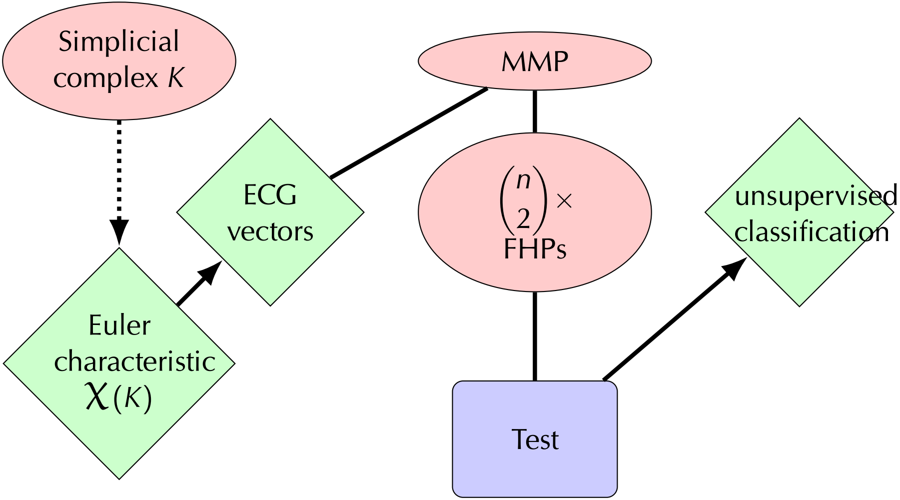


Discussion and the curse of dimensionality

- This approximation approach was first presented by Karnin *et. al.* in [4].
- No clear generalization to non-linear, non-separable, non-binary cases.
- Computing the singular vectors is still a very expensive operation.
- If the data lies uniformly in $\mathbb{R}^d$, it will tend to be concentrated in a sphere.



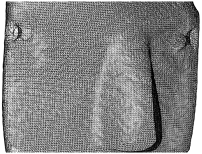
Where are we? (IV)

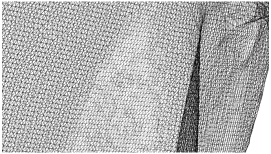


Offerings in Templo Mayor (Mexico City)

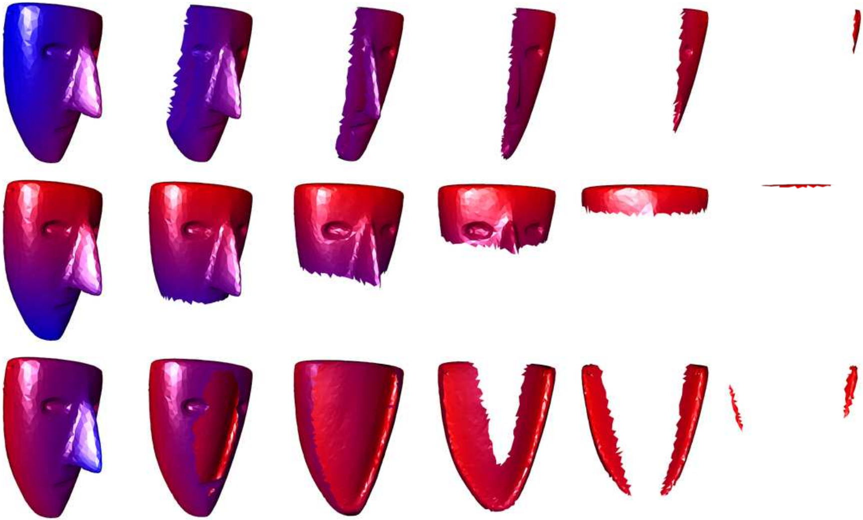
- We have 128 digital meshes of pre-Columbian masks.
- These are embedded in a $[-1, 1]^3$ cube with barycenter at origin.
- 8 families identified and a large unknown group.
- More details are explained by D. Jiménez in [3].

Set	No. of items	Set	No. of items
02	24	07	4
03	6	08	3
04	4	09	7
05	19	10	59
06	2	**TOTAL**	128

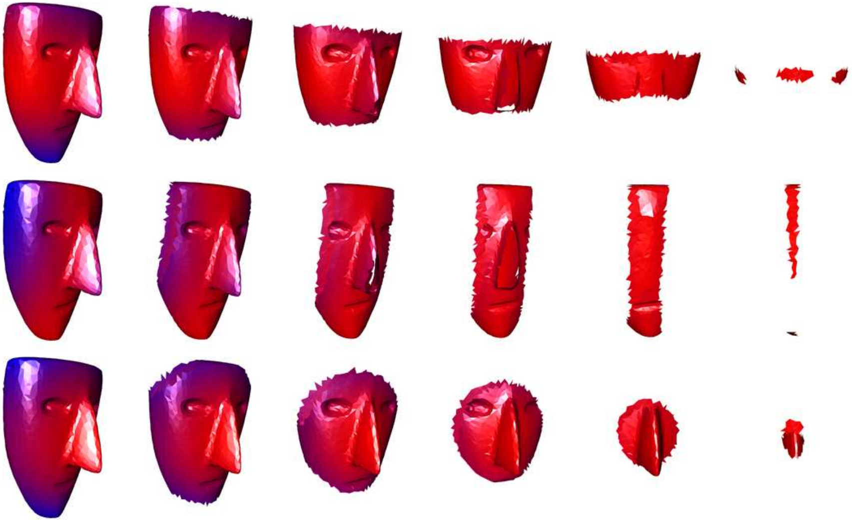




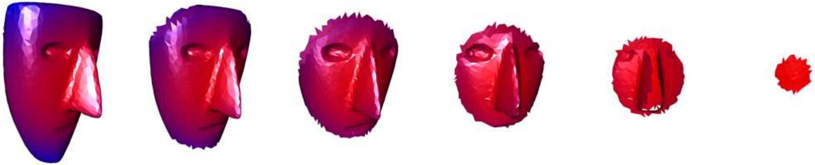
ECGs: Planes



ECGs: Cylinders



ECGs: Spheres



- Filter functions based on principal curvature values were also considered.
- These curvature filters yielded terrible results.
- Perhaps these was due to the fact the masks are extremely detailed.

Methodology for supervised SVMs

TRAINING: 8 clases

- Part of the items in families `02` and `05`.
- All the items in families `03, 04, 06, 07, 08`.
- The excluded items were those that D. Jiménez is skeptical about their current placing.

TEST

- The 128 masks in total.
- Our main goal is to classify items currently in `10`.

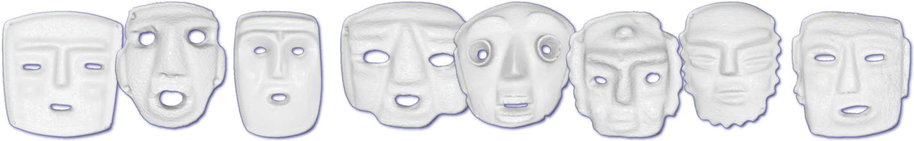
SVMS: 72 evaluaciones distintas

- Polynomial kernel $(\gamma\langle \mathbf{x}, \mathbf{y}\rangle + k)^{\text{deg}}$.
- Cost $C = 10$.
- Take the mode if it corresponds to at least 85% of answers.

Family 02

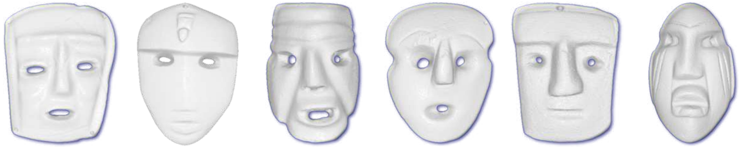
(a) 01 (b) 02 (c) 03 (d) 04 (e) 05 (f) 06 (g) 07 (h) 08

(i) 09 (j) 10 (k) 11 (l) 12 (m) 13 (n) 14 (o) 15 (p) 16

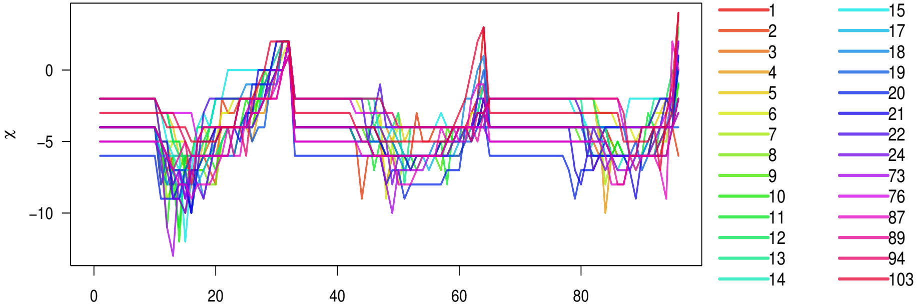


(q) 17 (r) 18 (s) 19 (t) 20 (u) 21 (v) 22 (w) 23 (x) 24

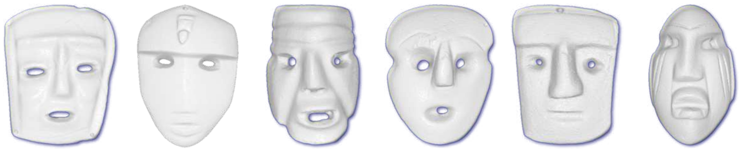
Fam 02 — Cylinders $T = 32$



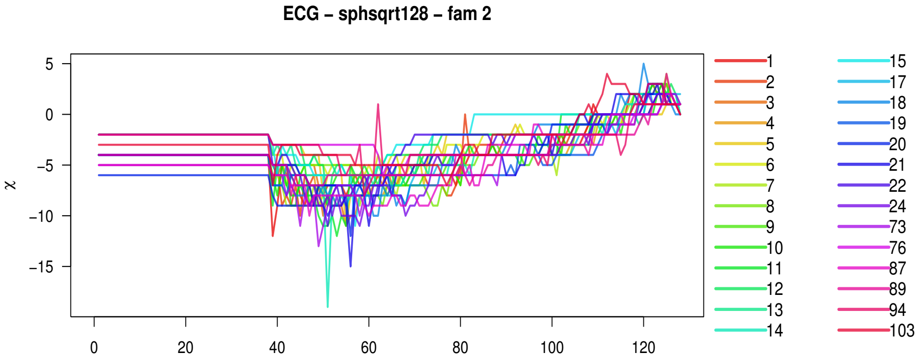
(a) 73 (b) 76 (c) 87 (d) 89 (e) 94 (f) 103



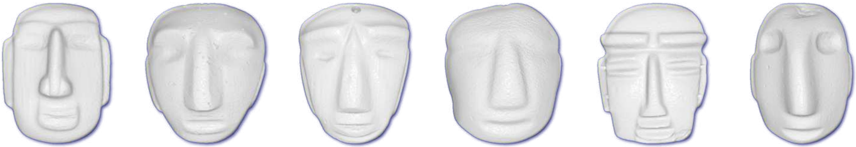
Fam 02 — Sphere $T = 128$



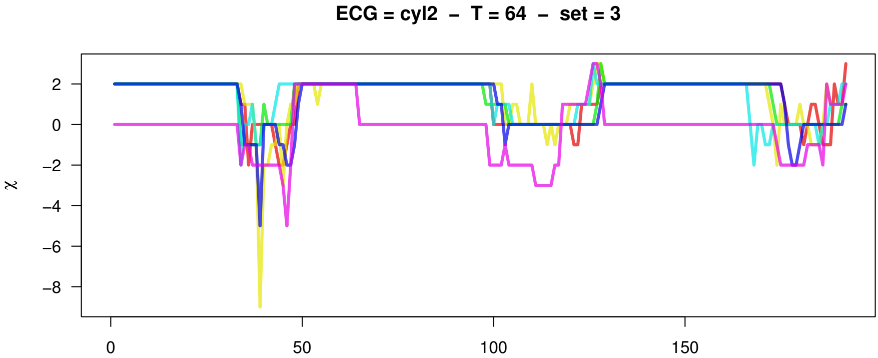
(a) 73 **(b)** 76 **(c)** 87 **(d)** 89 **(e)** 94 **(f)** 103



Family 03

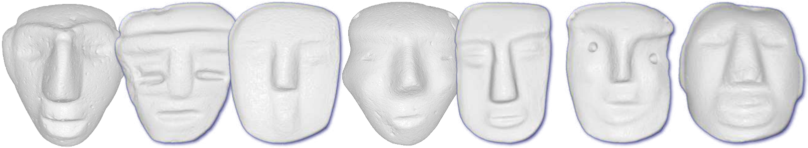


(a) 25 **(b)** 26 **(c)** 27 **(d)** 28 **(e)** 29 **(f)** 30



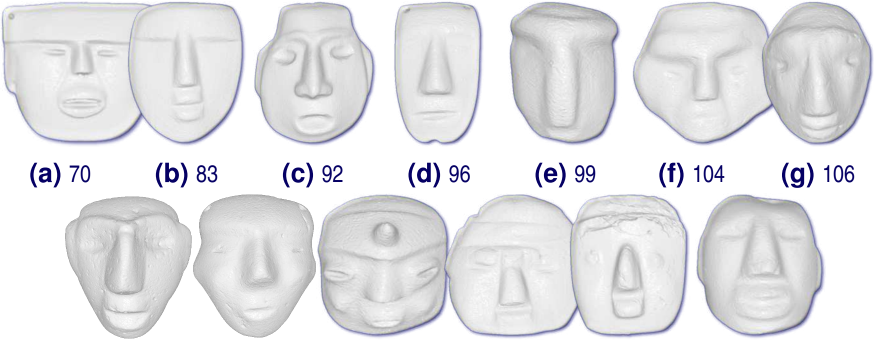
Cylinders T = 32

(a) 48 **(b)** 50 **(c)** 83 **(d)** 90 **(e)** 92 **(f)** 96 **(g)** 99 **(h)** 106



(i) 108 **(j)** 109 **(k)** 111 **(l)** 112 **(m)** 114 **(n)** 115 **(o)** 124

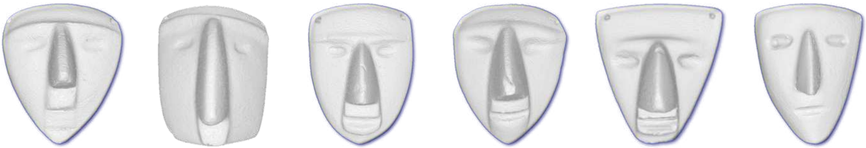
Spheres $T = 128$



(a) 70 (b) 83 (c) 92 (d) 96 (e) 99 (f) 104 (g) 106

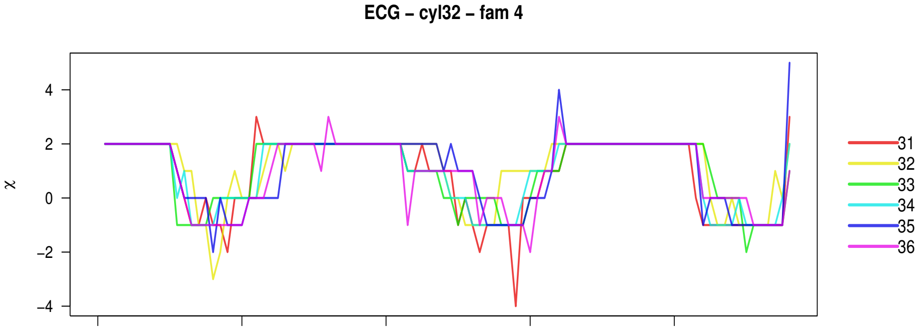
(h) 108 (i) 112 (j) 117 (k) 121 (l) 122 (m) 124

Family 04



(a) 31 **(b)** 32 **(c)** 33 **(d)** 34 **(e)** 35 **(f)** 36

Masks in the original set 04



Family 05


(a) 37


(b) 38


(c) 39


(d) 40


(e) 41


(f) 42


(g) 43


(h) 44


(i) 45


(j) 46


(k) 47


(l) 48


(m) 49


(n) 50

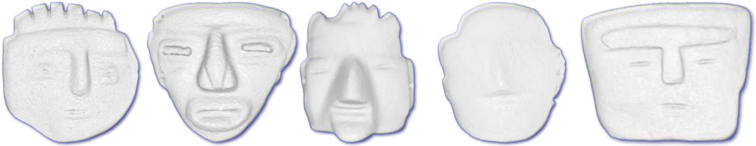

(o) 51


(p) 52


(q) 53

Cylinders $T = 32$

(a) 52 **(b)** 78 **(c)** 81 **(d)** 82 **(e)** 93 **(f)** 97



(g) 101 **(h)** 105 **(i)** 113 **(j)** 120 **(k)** 128

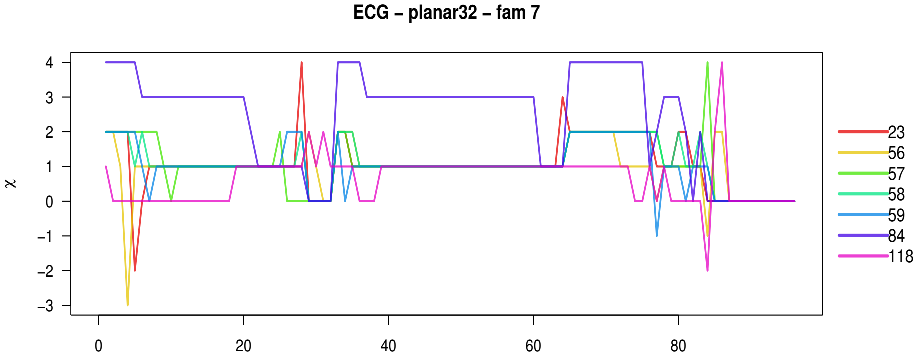
Family 07

(a) 56

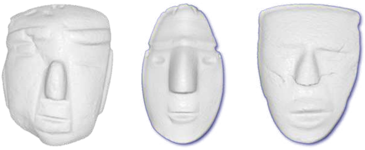
(b) 57

(c) 58

(d) 59

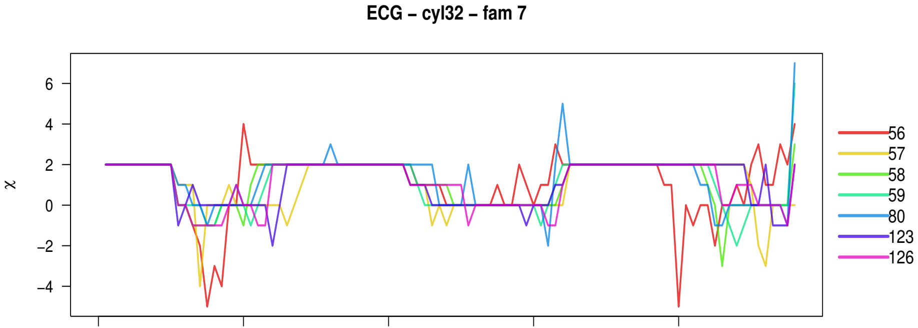


Cylinders $T = 32$

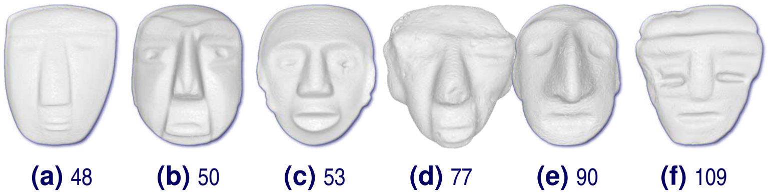


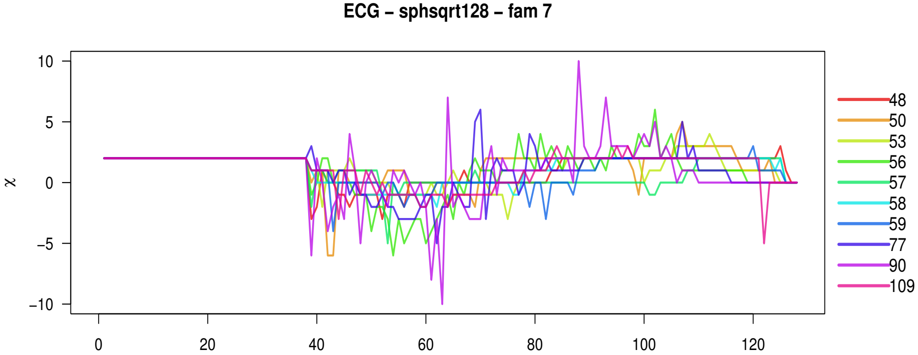
(a) 80 **(b)** 123 **(c)** 126

Masks assigned to Set 07 after running 72 polynomial SVMs



Spheres $T = 128$





Methodology: Unsupervised SVMs

- For each pair of oppositely labeled points $\mathbf{x}_i$ and $\mathbf{x}_j$, the unsupervised procedure yielded a splitting hyperplane $\mathbf{w}_{i,j}$.
- The unscaled margin defined by such hyperplane is

$$\bar{\theta}_{i,j} := \min_{1 \leq k \leq n} \left| \langle \mathbf{w}_{i,j}, \mathbf{x}_k \rangle \right| s_{i,j}.$$

- We want to see if the procedure can at least split accordingly two different training families.
- To reduce computation time, the dimension of the ECG vectors was reduced via standard PCA procedure.

Some information to compute

- We need to keep track if our data is affected by the high dimensionality.

$$\mu_{i,j} := \frac{1}{n} \sum_{1 \leq k \leq n} \left| \langle \mathbf{w}_{i,j}, \mathbf{x}_k \rangle \right| s_{i,j} \quad , \quad \sigma^2_{i,j} := \frac{1}{n} \sum_{1 \leq k \leq n} (|\langle \mathbf{w}_{i,j}, \mathbf{x}_k \rangle| s_{i,j} - \mu_{i,j})^2$$

$$M(\mu) := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \mu_{i,j} \quad , \quad \Sigma(\mu)^2 := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} (\mu_{i,j} - M(\mu))^2$$

$$M(\sigma^2) := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \sigma^2_{i,j} \quad , \quad \Sigma(\sigma^2)^2 := \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} (\sigma^2_{i,j} - M(\sigma^2))^2.$$

Family 02 vs. Families 05 and 09

- The algorithm was sensitive enough to distinguish masks with holes from masks without holes.
- It was sensitive to distinguish eye-holes and mouth-holes from other kind of holes.
- This result was replicated with different dimension reductions, different filters and different number of thresholds.
- The ECGs provide enough information.
- The pair of masks to provide the splitting hyperplane varied, however.

Family 03 vs. 04

- However, the procedure failed to distinguish apart two non-holed families.
- The mean and variance variances of distances suggest that the results are subject to high-dimensionality afflictions.

g	M	dim	$\bar{\Theta}$	$M(\mu)$	$\sqrt{\Sigma(\mu)}$	$M(\sigma^2)$	$\sqrt{\Sigma(\sigma^2)}$
planar	2	6	5.4	5.1	1.8	1.3	1.6
planar	2	12	5.4	5.2	1.6	0.9	0.4
cylinder	$\sqrt{2}$	6	7.9	7.4	2.5	0.7	0.5
cylinder	$\sqrt{2}$	12	8.2	8.2	1.9	0.6	0.3
cylinder	1	6	7.6	8.0	1.3	0.8	0.3
cylinder	1	12	7.8	7.2	2.0	2.0	1.5

- A similar result was encountered when comparing families 04 and 05.

Final words

- The computation of the ECG is an easy computation, linear in time. It can be quickly computed despite objects' large number of vertices.
- There is a large number of variable to tune.
- Several of the supervised procedures yielded sensible classifications, despite the limited amount of training data.
- A larger database, even if synthetic, might provide even more sensible assortments.
- More items might allow better results with an unsupervised approach.

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